Heat diffusion in a homogenous slab with an arbitrary periodical heat source: The case of the sinusoidal modulation function

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Abstract
The starting point in the study of the heat transfer and their applications is the solution of the heat diffusion equation with a particular boundary conditions kind congruent to the physical circumstances of the problem under consideration. Here, we calculate the solutions of the heat diffusion equation by means of the Green's functions technique, constrained by Dirichlet, Neumann and Robin's boundary conditions; making a comparison between the obtained solutions and discussing the behavior of the thermal response for every case. The calculations were done for an ideal homogonous solid sample, with a cylindrical symmetry, under the consideration of an arbitrary periodical heat source on one face of the sample. Finally, considering the particular case of a sinusoidal heat source, commonly used for the standard models in the field of the Photothermal science and techniques, the thermal response for each case of the three boundary conditions kind, is discussed.

Keywords: Diffusion equation, homogenous solid, photoacoustic technique, periodical function, sinusoidal modulation, thermal diffusivity, thermal wave.

Resumen
El punto de partida en el estudio de la transferencia de calor y sus aplicaciones es la solución de la ecuación de difusión de calor bajo condiciones de frontera particulares congruentes con las circunstancias físicas del problema bajo consideración. Aquí, calculamos las soluciones de la ecuación de difusión del calor, mediante la técnica de las funciones de Green, restringidas por condiciones de frontera del tipo Dirichlet, Neumann y Robin; realizando una comparación etre las soluciones obtenidas y discutiendo el comportamiento de la respuesta térmica en cada caso. Los cálculos fueron realizados para una muestra sólida ideal y homogénea, con una simetría cilíndrica y bajo la consideración de una fuente de calor arbitraria periódica en una cara de la muestra. Por último, considerando el caso particular de una fuente de calor sinusoidal, comúnmente utilizada por los modelos estándares en el campo de las ciencias y técnicas Fototérmicas, la respuesta térmica, en cada caso y bajo los tres tipos de condiciones de frontera, es discutida.

Palabras clave: Ecuación de difusión, sólida homogénea, técnica fotoacústica, función periódica, modulación sinusoidal, difusividad térmica, onda térmica.

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I. INTRODUCTION

Given a problem of heat transfer, many times the mathematical task of solving the heat diffusion equation (HDE) may be complicated or accessible, depending on the choice of the boundary conditions kind and the solution method to be used, among other no less important things like the symmetry, the number of degrees of freedom and the characteristics of the heat sources involved in the problem.

The HDE is a partial differential equation of fist order in time and second order in the spatial coordinates, therefore, is necessary to specify one condition in time, the initial condition, and two boundary conditions for each coordinate necessary in the description of the system, the boundary conditions of the problem. There are three kinds of boundary conditions generally used in problems of heat transfer. Dirichlet condition, also called first kind boundary condition, corresponds when the temperature surface is known. Neumann condition, or second kind boundary condition, corresponds when the heat flux is known. And Robin condition, also known as third kind boundary condition or newton law of cooling, corresponds to the existence of convection heating (or cooling) at the surface[1]. The choice of which boundary condition will be applied depends of the physical conditions existing at the boundaries of the medium. Of course, these three types of boundary conditions don't describe, nevertheless, all real conditions that occur in practice, such as body heating and cooling by radiation, the melting or freezing of bodies or complex heat transfer.

Since its introduction in 1828, Green's functions method have been widely used as a fundamental mathematical technique for solving boundary value problems in many areas of physics and engineering, of course, including the heat transfer area. Basically, in this method, for a given geometry, any field satisfying given source distributions and arbitrary initial conditions and boundary conditions can be constructed in the form of space and time integrals over the solution to the most elementary problem associated with the given geometry: that in which the source is in the form of a Dirac delta function in space and time, and the initial and boundary conditions are homogeneous everywhere[2, 3].

In this paper, the main goal is to solve the heat diffusion equation for the problem of an homogeneous slab with an arbitrary periodical heat source on one of its faces due to the light absorption, restricted to the most common boundary conditions (meaning Dirichlet, Neumann and Robin) used in heat flux problems, in order to obtain the differences among the thermal response for every case, which will be very useful in many experimental setups of the photothermal science and techniques.

II. THE GENERAL MATHEMATICAL MODEL

Consider a material sample with thickness l_s, on which a modulated light beam impinges uniformly on its normal direction. If $I(t,$r) denotes the absorbed power density by the sample, and keeping in mind the Beer-Lambert Law:[4]

$$I(t,\mathbf{r}) = (1-R)I_0\beta \exp(-\beta\mathbf{r}\cdot\hat{\mathbf{u}})\mathrm{T}(t). \qquad (1)$$

In Eq. (1), $\hat{\mathbf{u}}$ is a unitary vector normal to the incidence surface, $\mathrm{T}(t)$ is the modulation function (not necessarily a periodic function), and R, β are the reflexion coefficient and the optical absorption coefficient of the sample, respectively. It is a known fact that β is related to the wavelength λ of the incident beam, and with the imaginary part of the refraction index, named here as κ, through the next expression:

$$\beta = \frac{4\pi\kappa}{\lambda}. \qquad (2)$$

The absorbed power density will be transformed into heat by means of de-excitation processes. For crystalline and polycrystalline materials, some part of the energy absorbed is released by mechanical vibrations of the crystal lattice (this is the phonon contribution), and depending on the wavelength of the optical field, an excess of charge carriers could be produced (holes and electrons). These photogenerated charge carriers diffuse along the sample, recombining and possibly interacting among them and with phonons, producing extra contributions to the thermal relaxation of the sample. Some of these processes can be radiative, and some others non-radiative; this depends on the intensity of the absorbed power density, its wavelength and the excess charge density. In any case, an internal heat source $G(t,\mathbf{r})$, containing all the contributions of the light-heat conversion will appear[5]; but for the reach of the present paper, only the phonon contribution is considered. According to the heat diffusion equation

$$-\rho c\frac{\partial}{\partial t}\Theta(t,\mathbf{r}) + \nabla\cdot\left(k\nabla\Theta(t,\mathbf{r})\right) = G(t,\mathbf{r}) = (1-R)I_0\eta\beta\exp(-\beta\mathbf{r}\cdot\hat{\mathbf{u}})\mathrm{T}(t). \qquad (3)$$

Here: ρ, c y k are the volumetric mass density, the specific heat and the thermal conductivity of the sample, respectively, η is the light-into-heat energy conversion efficiency, and $\Theta(t,\mathbf{r})$ is the variation of the sample's temperature from the ambient temperature. The solutions of Eq. (3) are constrained by boundary conditions, which in the more general fashion are expressible as:

$$A_j\Theta(t,\mathbf{r})\Big|_{S_j} + B_j\frac{\partial}{\partial\hat{\mathbf{n}}_j}\Theta(t,\mathbf{r})\Big|_{S_j} = \Omega_j\,. \qquad (4)$$

In Eq. (4), A_j, B_j are constants, and Ω_j are functions that qualify the thermal contact of the sample with their surroundings and, if there is any, the losses through the sample's surfaces S_j. When it's necessary, the boundary conditions (4) can be substituted by regularity conditions over the temperatures variations or heat fluxes, to ensure stable solutions for Eq. (3). Since G becomes from the absorption of the power density I, it inherits the modulation of the light beam and so the temperature variation Θ are modulated too in the same way.

III. THE HEAT DIFFUSION EQUATION

This section deals with the problem of solving the heat diffusion equation for three of the most used boundary conditions, and some remarks on the behavior of the solution.

A. Solutions to the heat diffusion equation

Be a lineal, homogenous and isotropic medium, such that its geometry and the flux's direction of the incident light beam sustain a cylindrical symmetry, as Fig. 1 schematizes.

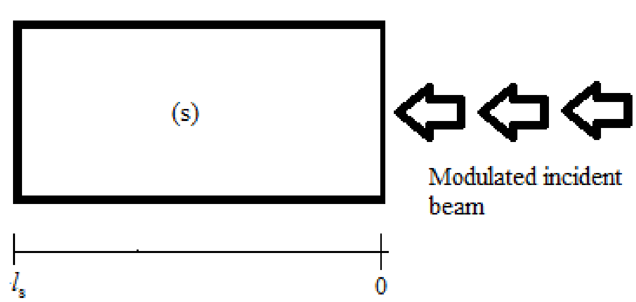

FIGURE 1. Scheme of the geometry of the system.

So, Eq. (3) reduces to Eq. (5), describing a mono dimensional heat diffusion process:

$$\frac{\partial^2}{\partial z^2}\Theta(t,z)-\frac{1}{\alpha_s}\frac{\partial}{\partial t}\Theta(t,z)=\frac{G(t,z)}{k_s}. \qquad (5)$$

Where $\alpha_s = k_s(\rho_s c_s)^{-1}$ is the thermal diffusivity of the sample (s) (in units of cm^2s^{-1}), describing the "speed" which the heat diffuses along the sample[a]. In addition, consider that the surroundings of the sample had an ambient temperature T_{amb}. Since the heat source (this is, the right member of Eq. (5)) had the same modulation than the optical power density, and for practical purposes the modulation function T(t) had an expansion in the Fourier basis, and so:

$$\begin{gathered} G(t,z)=\sum_m g_m(z)\exp(\mathrm{i}\omega_m t) \\ g_m(z)\equiv C_m\left[(1-R)I_0\eta\beta\exp(-\beta z)\right] \end{gathered}. \qquad (6)$$

Here: C_m are the expansion coefficients of T in the Fourier basis, $\omega_m \equiv 2m\pi f$ with f being the modulation frequency, and m an integer. Then, the main goal depends on the solution of the next boundary value problem:

$$\frac{\partial^2}{\partial z^2}\theta(\omega',z)-\sigma_s^2\theta(\omega',z)=\sqrt{2\pi}\sum_m g_m(z)\delta(\omega'-\omega_m)$$

$$A_f\theta(\omega',z)-B_f\frac{\partial}{\partial z}\theta(\omega',z)\bigg|_{z=0}=\Omega_f \quad ; \quad A_r\theta(\omega',z)+B_r\frac{\partial}{\partial z}\theta(\omega',z)\bigg|_{z=l_s}=\Omega_r. \qquad (7)$$

In expressions (7), θ is the Unitary Fourier Transform (respect to time) of the temperature variation Θ, $\sigma_s \equiv (1+i)/\mu_s$ is defined by means of the thermal diffusion length[6] $\mu_s \equiv (2\alpha_s/\omega')^{1/2}$, and the indexes {f, r} label the front (illuminated) and rear (non-illuminated) surfaces of the sample. At this work, homogenous boundary conditions were considered, *i.e.* $\Omega_f = \Omega_r = 0$. Using the Green's functions technique, [3, 7] the solution to Eq. (7), can be written as:

$$\theta(\omega',z)=\sqrt{2\pi}\sum_m\left(\int_0^{l_s}\mathrm{K}(\omega',z,z')g_m(z')\mathrm{d}z'\right)\delta(\omega'-\omega_m). \qquad (8)$$

Where: K is the Green function satisfying:

$$\frac{\partial^2}{\partial z^2}\mathrm{K}(\omega',z,z')-\sigma_s^2\mathrm{K}(\omega',z,z')=\delta(z-z'). \qquad (9)$$

Under boundary conditions equivalent to (7), and:

$$\lim_{z\to z'}\Delta\mathrm{K}=0 \quad ; \quad \lim_{z\to z'}\Delta\left(\frac{\partial}{\partial z}\mathrm{K}\right)=1. \qquad (10)$$

So, the family of solutions of Eq. (9) is given by:

$$\mathrm{K}(\omega',z,z')=\begin{cases} A_> \exp(\sigma_s z)+B_> \exp(-\sigma_s z)\,;\, z>z' \\ A_< \exp(\sigma_s z)+B_< \exp(-\sigma_s z)\,;\, z<z' \end{cases}. \qquad (11)$$

B. Solutions under Dirichlet boundary conditions

In this case $A_f = A_r = 1$ y $B_f = B_r = 0$. The restriction implies the continuity of the temperature distribution across the interfacial surfaces, therefore, the Green function for this kind of boundary conditions is written as follows:

$$\mathrm{K}_D(\omega',z,z')=\begin{cases} \dfrac{\mathrm{Sinh}\,\sigma_{ms}(z-l_s)\,\mathrm{Sinh}\,\sigma_{ms}z'}{\sigma_{ms}\mathrm{Sinh}\,\sigma_{ms}l_s}\,;\, z>z' \\ \dfrac{\mathrm{Sinh}\,\sigma_{ms}z\,\mathrm{Sinh}\,\sigma_{ms}(z'-l_s)}{\sigma_{ms}\mathrm{Sinh}\,\sigma_{ms}l_s}\,;\, z<z' \end{cases}. \qquad (12)$$

For demonstration purposes, in this paper will be considered that the only contribution to the internal heat source is the phonon contribution (as we mentioned at the beginning) and from Eq. (8), we obtain the response on the frequency domain to be:

$$\theta_D(\omega',z)=\sum_m\frac{A_m r_s}{(r_s^2-1)}\left[H(\omega',z-l_s)F(\omega',z)-H(\omega',z)F(\omega',z-l_s)\exp(-\beta l_s)\right]. \qquad (13)$$

Where:

$$H(\omega',\varsigma)\equiv\frac{\mathrm{Sinh}\,\sigma_s\varsigma}{\sigma_s\mathrm{Sinh}\,\sigma_s l_s}$$

$$F(\omega',\varsigma)\equiv 1-\exp(-\beta\varsigma)\left[r_s\mathrm{Sinh}\,\sigma_s\varsigma+\mathrm{Cosh}\,\sigma_s\varsigma\right]. \qquad (14)$$

$$r_s\equiv\frac{\beta}{\sigma_s} \quad ; \quad A_m=\sqrt{2\pi}\,\frac{(1-R)I_0\eta}{\kappa_s}C_m$$

By means of the Inverse Unitary Fourier Transform of Eq. (13), the temperature distribution (under Dirichlet boundary conditions) in time domain is:

$$\Theta_D(t,z)=\sum_m\frac{\theta_D(\omega_m,z)}{\sqrt{2\pi}}\exp(\mathrm{i}\omega_m t). \qquad (15)$$

[a] This quantity is the analog to the diffusion coefficient in mass diffusion processes, as can be read it in: *The Mathematics of Diffusion*, J. Crank, Second edition, Oxford University Press (New York, 1975), pp 8-10.

C. Solutions under Neumann boundary conditions

In this second case, $A_f = A_r = 0$ y $B_f = B_r = k_s$, and so, the continuity of the heat flux across the interfacial surfaces is guarantee. In such case, the Green function is the following:

$$\mathrm{K}_{\mathrm{N}}(\omega',z,z') = \begin{cases} -\dfrac{\mathrm{Cosh}\,\sigma_{ms}(z-l_s)\,\mathrm{Cosh}\,\sigma_{ms}z'}{\sigma_{ms}\mathrm{Sinh}\,\sigma_{ms}l_s}\,;\; z > z' \\ -\dfrac{\mathrm{Cosh}\,\sigma_{ms}z\,\mathrm{Cosh}\,\sigma_{ms}(z'-l_s)}{\sigma_{ms}\mathrm{Sinh}\,\sigma_{ms}l_s}\,;\; z < z' \end{cases}. \quad (16)$$

Substituting Eq. (16) into Eq. (8) we obtain the response on the frequency domain to be:

$$\theta_{\mathrm{N}}(\omega',z) = \sum_m \frac{A_m r_s}{(r_s^2-1)}\left[M(\omega',z-l_s)S(\omega',z) - M(\omega',z)S(\omega',z-l_s)\exp(-\beta l_s)\right]. \quad (17)$$

Where:

$$M(\omega',\zeta) \equiv \frac{\mathrm{Cosh}\,\sigma_s\varsigma}{\sigma_s\mathrm{Sinh}\,\sigma_s l_s}$$
$$S(\omega',\zeta) \equiv r_s - \exp(-\beta\zeta)\left[\mathrm{Sinh}\,\sigma_s\varsigma + r_s\mathrm{Cosh}\,\sigma_s\varsigma\right] \quad (18)$$

In similar way, applying the Inverse Unitary Fourier Transform to Eq. (17), the temperature distribution (under Neumann boundary conditions) in time domain is given then by:

$$\Theta_{\mathrm{N}}(t,z) = \sum_m \frac{\theta_{\mathrm{N}}(\omega_m,z)}{\sqrt{2\pi}}\exp(\mathrm{i}\omega_m t). \quad (19)$$

D. Solutions under Robin boundary conditions

This third case, also known as impedance boundary conditions, $A_r = A_f = h$, and $B_r = B_f = k_s$. In this kind of boundary conditions, h represents the overall heat exchange coefficient, and depends on the surrounding medium as well the physical properties of the sample. So, the homogenous Robin boundary condition states that the total heat flux is conserved, taking into account the conductive, convective and radiative heat fluxes.

Of course, the Green function expected for this kind of boundary conditions will be more complicate, and after some calculations is written as bellow:

$$\mathrm{K}_{\mathrm{R}}(\omega',z,z') = \begin{cases} \dfrac{\left[\mathrm{Cosh}\,\sigma_s(z-l_s) - e_s\mathrm{Sinh}\,\sigma_s(z-l_s)\right]\left[\mathrm{Cosh}\,\sigma_s z' + e_s\mathrm{Sinh}\,\sigma_s z'\right]}{\sigma_s[(1\text{-}e_s^2)\mathrm{Sinh}\,\sigma_s l_s - 2e_s\mathrm{Cosh}\,\sigma_s l_s]}\,;\; z > z' \\ \dfrac{\left[\mathrm{Cosh}\,\sigma_s z + e_s\mathrm{Sinh}\,\sigma_s z\right]\left[\mathrm{Cosh}\,\sigma_s(z'-l_s) - e_s\mathrm{Sinh}\,\sigma_s(z'-l_s)\right]}{\sigma_s[(1\text{-}e_s^2)\mathrm{Sinh}\,\sigma_s l_s - 2e_s\mathrm{Cosh}\,\sigma_s l_s]}\,;\; z < z'. \end{cases} \quad (20)$$

In the past equation, $e_s \equiv h(k_s\sigma_s)^{-1}$. If we considered that the Biot number $\mathrm{Bi}_s \equiv hl_s k_s^{-1}$ is a simple index of the ratio of the heat transfer resistance of and at the surface of the sample (and therefore qualifies the ability of the sample to exchange heat through their surfaces), it is possible to rewrite e_s as:

$$e_s = \frac{\mathrm{Bi}_s\mu_s}{\sqrt{2}l_s}\exp(-\mathrm{i}\pi/4). \quad (21)$$

The coefficient e_s is a dimensionless quantity, being a function not only of the solid sample and its surroundings, but also a function of the modulation frequency, diminishing at the time that the modulation frequency gets larger.

Substituting Eq. (20) into Eq. (8) we obtain the response on the frequency domain to be:

$$\theta_{\mathrm{R}}(\omega',z) = \sum_m \frac{A_m r_s}{(r_s^2-1)}\left\{\frac{\left[\left(W_{(-)}\circ V\right)(\omega',z-l_s)\right]Q_{(+)}(\omega',z) - \left[\left(W_{(+)}\circ V\right)(\omega',z)\right]Q_{(-)}(\omega',z-l_s)}{\sigma_s[(1\text{-}e_s^2)\mathrm{Sinh}\,\sigma_s l_s - 2e_s\mathrm{Cosh}\,\sigma_s l_s]}\right\}. \quad (22)$$

In Eq. (22), $\circ$, denotes the function composition operator, and the following definitions were used:

$$Q_{(\pm)}(\omega',\zeta) \equiv W_{(\pm)}(r_s,1) - \exp(-\beta\zeta)\left[r_s\left(W_{(\pm)}\circ V\right)(\omega',\zeta) + \left(W_{(\pm)}\circ U\right)(\omega',\zeta)\right]$$

$$W_{(\pm)}(X,Y) \equiv X \pm e_s Y$$

$$U(\omega',\zeta) \equiv \left(\mathrm{Sinh}\,\sigma_s\zeta, \mathrm{Cosh}\,\sigma_s\zeta\right) \;;\; V(\omega',\zeta) \equiv \left(\mathrm{Cosh}\,\sigma_s\zeta, \mathrm{Sinh}\,\sigma_s\zeta\right). \quad (23)$$

The temperature distribution, under Robin boundary conditions, in time domain is written finally as:

$$\Theta_{\mathrm{R}}(t,z) = \sum_m \frac{\theta_{\mathrm{R}}(\omega_m,z)}{\sqrt{2\pi}}\exp(\mathrm{i}\omega_m t). \quad (24)$$

IV. SPECIAL CASE: SINUSOIDAL MODULATION

This section presents the theoretical results for the most common modulation function used in experimental setups: The sinusoidal modulation. This kind of modulation is historically important because was the modulation used by Rosencwaig and Gersho[8] in their earliest papers where the photoacoustic effect was explained for the first time, and since then, it has being used by the majority of the researchers in posterior models. Since the sinusoidal modulation is simple, offers solutions to the heat diffusion equation relatively easy to handle in minimum square fitting processes to experimental data; however, this modulation is an approximation to the actual experimental conditions. Frequently, a mechanical modulator (chopper) of variable speed is used to modulate the continuous light beam emerging from a light source, and in this manner, what we have in reality is a train of square pulses *i.e.*, a square wave modulation[9], which will be treated in a subsequent work.

So, if a sinusoidal modulation is used, by means of the orthogonality relationship of the Fourier basis, for Eq. (6) it must be considered that:

$$C_m = \begin{cases} 1 \quad ; \quad m = -1,1 \\ \\ 0 \quad ; \quad m \neq -1,1 \end{cases}. \tag{25}$$

And therefore, in Eqs. (14, 18) and (23) the sum only covers the harmonics m = -1 and 1. For a better theoretical analysis, the calculations were done by considering the relative frequency ν, and relative position z^*, defined as follows:

$$\nu \equiv \frac{f}{f_c} \quad ; \quad z^* \equiv \frac{z}{l_s}. \tag{26}$$

In Eq. (26), the quantity $f_c \equiv \alpha_s(\pi l_s^2)^{-1}$ is known as the characteristic frequency, and represents the value of the modulation frequency at which the thermal diffusion length equals the thickness of the sample. The characteristic frequency is strongly related to the definition of the thermal regimes. It says that a sample is thermally thin when its thickness is much smaller than its thermal diffusion length, *i.e.*, $f \ll f_c$. On the contrary, it says that a sample is thermally thick when its thickness is much greater than its thermal diffusion length, *i.e.*, $f_c \ll f$. The use of dimensionless variables allows reproduce the behavior of all solids, since the thermal and geometrical characteristics are not explicit.

Figs. 2 and 3 show the amplitude and phase of the temperature variations θ_D, and θ_N as functions of relative position z^* and frequency ν, for the boundary conditions of Dirichlet and Neumann, respectively. In the calculations values of βl_s = 300 were considered (*i.e.*, the sample is considered as optically opaque). In Fig. 2, θ_D is null at z^* = 0, for all values of ν, as is expected from the boundary condition, reaching a maximum in the interior, and decreasing as function of ν for a given z^*. On the other hand, in Fig. 3, θ_N increases quickly when ν goes to zero for each value of z^*, and θ_N remains practically unchanged with z^* for each value of ν. These results reflect accurately the restrictions imposed by the Dirichlet and Neumann boundary conditions, that is, the continuity of the temperature distribution and the heat flux across the interfaces, respectively.

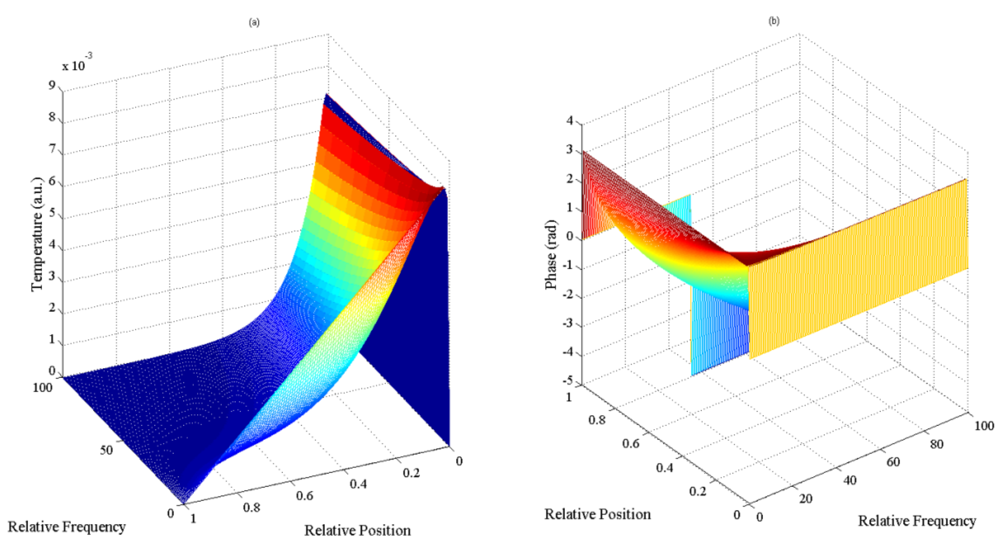

FIGURE 2. Calculation of: (a) Amplitude of the temperature variations, and (b) Phase of the temperature variations, as function of relative position and frequency. Dirichlet boundary conditions were considered.

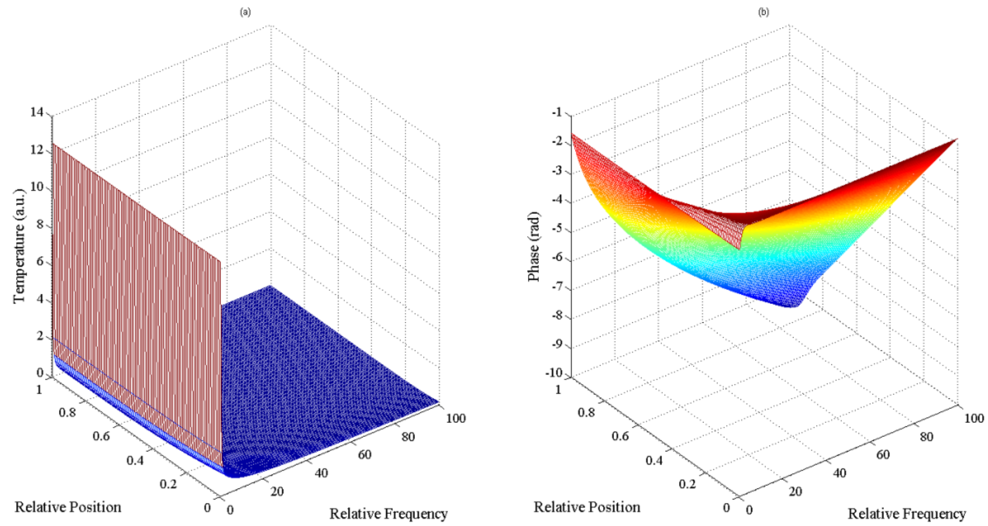

FIGURE 3. Calculation of: (a) Amplitude of the temperature variations, and (b) Phase of the temperature variations, as function of relative position and frequency. Neumann boundary conditions were considered.

Figure 4 shows the amplitude and phase of the temperature variation θ_R as a function of relative position z^* and frequency v, for Robin boundary condition. Here, values of $\beta l_s = 300$, and $Bi_s = 0.5$ were considered. Again, sinusoidal modulation was used for the theoretical calculation. In this case, the consideration of the convective heat flux leads to a greater change in θ_R, in comparison with the results obtained with the Neumann boundary condition. θ_R increases more quickly when v goes to zero for each value of z^*, and θ_R increases when z^* goes to zero for each value of v.

The value for the Biot number used here for theoretical calculations could be considered as a huge one. This value was chosen only to show the influence of large Biot number on the solutions of the heat diffusion equation. This influence rapidly diminishes for relative small values for the Biot number (meaning $Bi_s \leq 10^{-2}$), however, manipulating the geometric parameters and the surrounding medium it is always achievable large values for Bi_s.

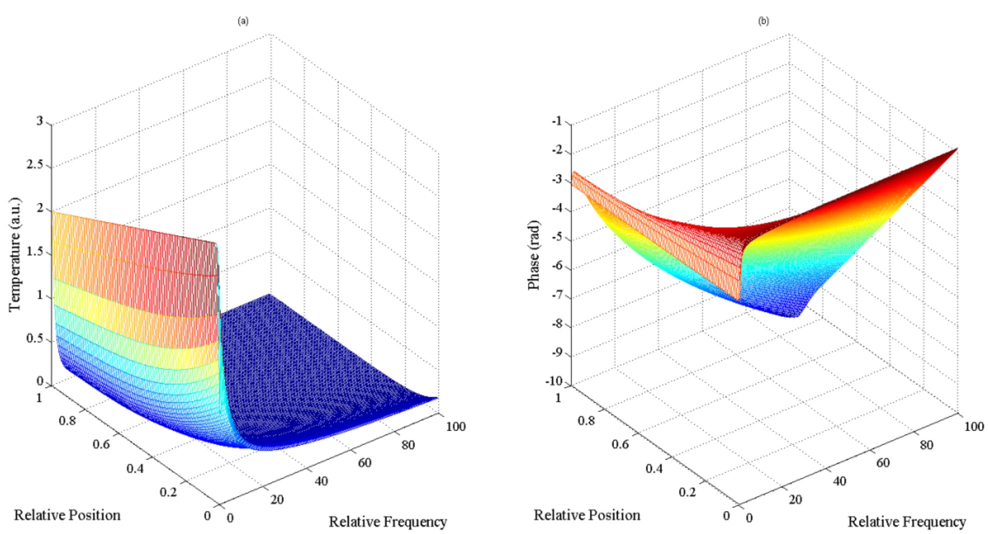

FIGURE 4. Calculation of: (a) Amplitude of the temperature variations, and (b) Phase of the temperature variations, as function of relative position and frequency. Robin boundary conditions were considered.

In fact, large values of Bi_s are related to the thermally thick regime in studies of the transient behavior for continuous illumination,[10, 11] *i.e.*, constant modulation, and so is an important factor in the transient behavior in the case of modulated illumination.[12] Here, because the general modulation is used in the mathematical model, the thermally regimens are (in the case of Robin boundary conditions) defined by both, the characteristic frequency and the Biot number, because they affect the Green's function (see Eq. (19)) through the coefficients e_s and σ_s.

Now, how much influence does the Bi_s number in the thermal response? The answer is not obvious from the Eqs. (19) to (24), because the influence of Bi_s appear through the coefficient e_s, where the size of Bi_s is in somehow modulated by the thermal diffusion length, which is a function of the modulation frequency. In Fig. 5, different values of Bi_s number are used for the calculation of the temperature distribution (at $z^* = 0.5$), and the results are compared to the solutions under Neumann boundary condition, in the relative frequency domain.

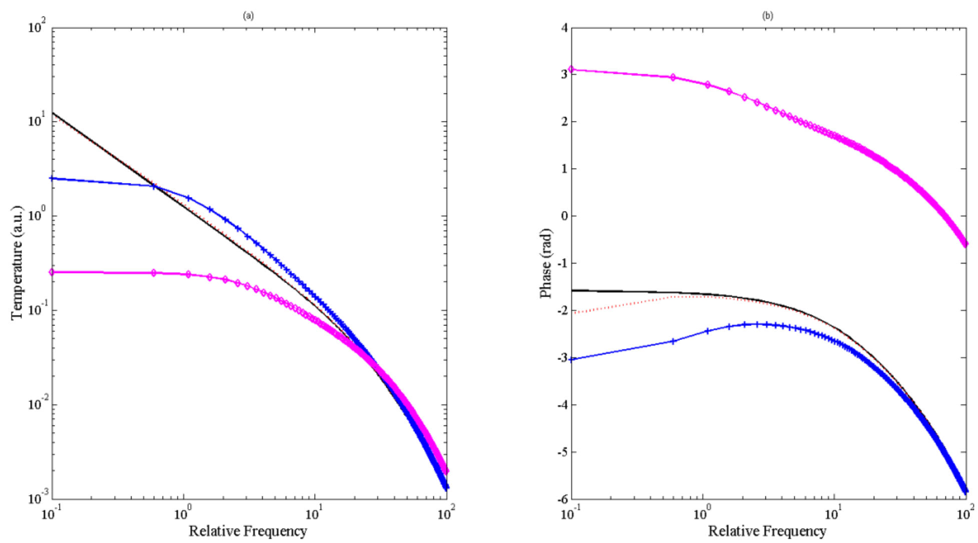

FIGURE 5. Comparison of the behavior of: (a) Amplitude and (b) Phase of the temperature variations as function of relative frequency, for sinusoidal wave modulation. The black line represents the solutions under Neumann boundary conditions. The calculations under Robin boundary conditions were performed for different values of Bi_s: 0.05(red dotted line), 0.5 (blue line) and 5 (pink line).

It can be viewed, from Fig. 5, that the solutions are quite much alike for small values of the Biot number, and as the values of Bi_s increases, the differences increases too (especially in the phase). This is the expected behavior since large values of Bi_s are related to a greater contribution of the convection and radiation terms to the heat flux. However, for sufficient larges values of the relative frequency, the calculations tend to equalize, supporting the earlier comments on the modulation of the influence of the Bi_s number by the relative frequency.

V. CONCLUSIONS

The comparison among the results obtained for the thermal response determined by the application of Dirichlet, Neumann and Robin boundary conditions show that the selection of a particular kind of boundary condition is definitive in the predicted behavior of the thermal response, being the calculated phase difference the clearest visualization of this influence. Each boundary condition should be consistent to a particular problem to be solved, since each of them demands physical conditions that can be fulfilled by one type of boundary conditions. Of course, there are other kinds of boundary conditions (like mixed boundary conditions, usually used in radial heat flux) consistent to different geometrical, environmental and physical conditions. However, the general expression for the Green function given in Eq.(11) are suitable to be used, as long the parabolic form of the heat diffusion equation is maintained, and from Eq. (8) other contributions, in addition to the phonon contribution can be included in the heat diffusion process. Also, is remarkable the relationship between the values of the Biot number and the characteristic frequency (through the relative modulation frequency) in the thermal response of a studied sample, when modulated excitation is used in problems where are considered convection and/or radiation contributions to the total heat flux, changing the definitions of the thermal regimes for a solid sample.

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